

$$1. (a) u = R(r)\theta(\theta)$$

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$$u(r,0) = 0$$

$$u(r,\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

$$u(a,\theta) = f(\theta)$$

$$u(r,\theta) \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r R' \theta) + \frac{1}{r^2} R \theta'' = 0$$

$$\text{i.e. } \frac{1}{r} R' \theta + R'' \theta + \frac{1}{r^2} R \theta'' = 0$$

$$r R' \theta + r^2 R'' \theta + R \theta'' = 0$$

$$\frac{r R'}{R} + \frac{r^2 R''}{R} + \frac{\theta''}{\theta} = 0 = \lambda^2$$

$$\text{say } \frac{r R'}{R} + \frac{r^2 R''}{R} = -\frac{\theta''}{\theta} = \lambda^2 \quad (\lambda^2 \text{ since want oscillatory in } r)$$

$$\text{i.e. } r^2 R'' + r R' - \lambda^2 R = 0$$

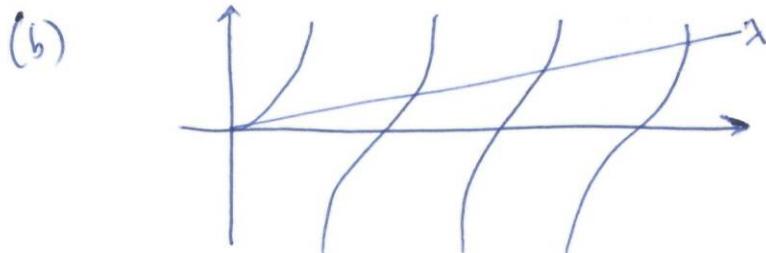
$$\theta'' + \lambda^2 \theta = 0$$

$$\text{Solving } \theta : \theta = A \cos \lambda \theta + B \sin \lambda \theta$$

$$\text{b.c. } \theta(0) = A = 0 \Rightarrow A = 0 \Rightarrow \theta = B \sin \lambda \theta$$

$$\theta(\pi) = B \sin \lambda \pi$$

$$\theta'(\pi) = \frac{B}{\lambda} \cos \lambda \pi \quad \left. \vphantom{\theta(\pi)} \right\} \Rightarrow \tan \lambda_n \pi = \lambda_n$$



$$(c) r^2 R'' + r R' - \lambda^2 R = 0$$

Look for solns $R = r^\alpha$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + r \alpha r^{\alpha-1} - \lambda^2 r^\alpha = 0$$

$$\alpha(\alpha-1) + \alpha - \lambda^2 = 0$$

$$\alpha^2 - \alpha + \alpha - \lambda^2 = 0 \Rightarrow \alpha = \pm \lambda \Rightarrow R = r^{\pm \lambda}$$

$$\text{need } u(r,\theta) \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$\Rightarrow R = r^{-\lambda}$$

$$\Rightarrow u = \theta R = \sum_{n=1}^{\infty} A_n \sin \lambda_n \theta r^{-\lambda_n}$$

$$u = \sum_{n=1}^{\infty} A_n \sin \lambda_n \theta r^{-\lambda_n}$$

We know $u(a, \theta) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n \theta) a^{-\lambda_n} = f(\theta)$

by orthogonality $A_n a^{-\lambda_n} \int_0^{\pi} \sin^2(\lambda_n \theta) d\theta = \int_0^{\pi} \sin \lambda_n \theta \cdot f(\theta) d\theta$

$$\text{i.e. } A_n a^{-\lambda_n} = \frac{\int_0^{\pi} \sin(\lambda_n \theta) \cdot f(\theta) d\theta}{\int_0^{\pi} \sin^2(\lambda_n \theta) d\theta}$$

$$\Rightarrow u = \sum_{n=1}^{\infty} \frac{a^{\lambda_n} \int_0^{\pi} \sin(\lambda_n \theta) \cdot f(\theta) d\theta}{\int_0^{\pi} \sin^2(\lambda_n \theta) d\theta} \sin(\lambda_n \theta) r^{-\lambda_n}$$

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} \dots \dots \dots \sin(\lambda_n \theta) (-\lambda_n) r^{-\lambda_n-1}$$

$$\frac{\partial u}{\partial r}(a, \theta) = \sum \frac{a^{\lambda_n} \int_0^{\pi} \sin(\lambda_n \theta) \cdot f(\theta) d\theta}{\int_0^{\pi} \sin^2(\lambda_n \theta) d\theta} \sin(\lambda_n \theta) (-\lambda_n) a^{-\lambda_n} a^{-1}$$

$$= - \sum \frac{\lambda_n}{a} \frac{\int_0^{\pi} f(\phi) \sin \lambda_n \phi d\phi}{\int_0^{\pi} \sin^2 \lambda_n \phi d\phi} \sin \lambda_n \theta .$$

2. $x^2 y'' + 3xy' + (1+x)y = 0$

Can also write as $y'' + \frac{3}{x}y' + \frac{1+x}{x^2}y = 0$

hence in usual notation, $p(x) = \frac{3}{x}$
 $q(x) = \frac{1+x}{x^2}$ } both singular

and $x p(x) = 3$
 $x^2 q(x) = 1+x$ } both analytic \Rightarrow RSP.

$y(x) = x^c \sum_{k=0}^{\infty} a_k x^k$ ie. $y(x) = \sum_{k=0}^{\infty} a_k x^{k+c}$

$\Rightarrow y' = (k+c) \sum_{k=0}^{\infty} a_k x^{k+c-1}$

$y'' = (k+c-1)(k+c) \sum_{k=0}^{\infty} a_k x^{k+c-2}$

$\Rightarrow \sum (k+c-1)(k+c) a_k x^{k+c} + \sum 3(k+c) a_k x^{k+c} + \sum a_k x^{k+c} + \sum a_k x^{k+c+1} = 0$

$= \sum [(k+c-1)(k+c) + 3(k+c) + 1] a_k x^{k+c} + \sum a_{k-1} x^{k+c} = 0$

ie. $\sum [(k+c-1)(k+c) + 3(k+c) + 1] a_k + a_{k-1} x^{k+c} = 0$

$\Rightarrow [(k+c-1)(k+c) + 3(k+c) + 1] a_k + a_{k+1} = 0$

ie. $[k^2 + 2ck + c^2 - k - c + 3k + 3c + 1] a_k + a_{k+1} = 0$

ie. $[k^2 + 2ck + c^2 + 2k + 2c + 1] a_k + a_{k+1} = 0$

ie. $[k+c+1]^2 a_k + a_{k+1} = 0$

$(k+c+1)^2 = k^2 + kc + k + ck + c^2 + c + k + c + 1$
 $= k^2 + 2kc + c^2 + 2k + 2c + 1$

$\Rightarrow a_k = \frac{-a_{k-1}}{[k+c+1]^2}$ recurrence.

$k=0: [c+1]^2 a_0 = 0 \Rightarrow \underline{c = -1, -1}$ indicial.

$$a_k = \frac{-a_{k-1}}{(k+c+1)^2}$$

$$a_1 = \frac{-a_0}{(c+2)^2}$$

$$a_2 = \frac{-a_1}{(c+3)^2} = \frac{a_0}{(c+2)^2(c+3)^2}$$

$$\Rightarrow a_k = \frac{(-1)^k a_0 [(c+1)!]^2}{([c+k+1]!)^2}$$

$$\Rightarrow a_k = \frac{(-1)^k [(c+1)!]^2}{[c+k+1]!^2}$$

$$\Rightarrow y_1 = \sum_{k=0}^{\infty} a_k x^{k+c} \quad \rightarrow$$

$$y_2 = \frac{d}{dc} \sum_{k=0}^{\infty} \frac{(-1)^k [(c+1)!]^2 x^{k+c}}{[c+k+1]!^2}, \text{ evaluated at } c=-1.$$

$$= \sum \frac{\ln x (-1)^k [(c+1)!]^2 x^{k+c}}{[c+k+1]!^2} + \sum (-1)^k x^{k+c} \frac{d}{dc} \left[\frac{[(c+1)!]^2}{[c+k+1]!^2} \right]$$

use log. diff.

$$\frac{d}{dc} \ln a_k = \frac{1}{a_k} \frac{da_k}{dc}$$

$$\ln a_k = \ln \left(\frac{(-1)^k [(c+1)!]^2}{[c+k+1]!^2} \right) = 2 \ln [(c+1)!] - 2 \ln [c+k+1]!$$

$$\frac{d}{dc}$$

etc.

3. (a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

has polynomial solution $P_n(x)$

$\Rightarrow (1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \dots (1)$

also $(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \dots (2)$

$P_m \cdot (1) - P_n \cdot (2) :$

$(1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m$
 $- (1-x^2)P_m''P_n + 2xP_m'P_n - m(m+1)P_nP_m = 0$

integrating:

$\int_{-1}^1 (1-x^2)P_n''P_m - (1-x^2)P_m''P_n - 2xP_n'P_m + 2xP_m'P_n + [n(n+1) - m(m+1)]P_nP_m dx = 0$

by parts:

$[(1-x^2)P_mP_n']_{-1}^1 - \int_{-1}^1 [(-2x)P_m + (1-x^2)P_m']P_n' dx$

$- [(1-x^2)P_nP_m']_{-1}^1 + \int_{-1}^1 [(-2x)P_n + (1-x^2)P_n']P_m' dx$

$- [2xP_mP_n]_{-1}^1 + \int_{-1}^1 [2P_m + 2xP_m']P_n dx$

$+ [2xP_nP_m]_{-1}^1 + \int_{-1}^1 [2P_n + 2xP_n']P_m dx$

$+ [n(n+1) - m(m+1)] \int_{-1}^1 P_nP_m dx = 0$

$\Rightarrow [n(n+1) - m(m+1)] \int_{-1}^1 P_nP_m dx = 0 \Rightarrow \int_{-1}^1 P_n(x)P_m(x) dx = 0, m \neq n.$

(b) $\exp\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum t^n J_n(x)$

$d/dx: \frac{1}{2}\left(t - \frac{1}{t}\right) \exp\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum t^n J_n'(x)$

(iii) $2\pi J_0(x)$

$$\exp\left[\frac{1}{2}x(e^{i\theta} - e^{-i\theta})\right] = \sum e^{in\theta} J_n(x)$$

$$\text{"} \\ \exp[ix\sin\theta] = \cos[x\sin\theta] + i\sin[x\sin\theta].$$

Real part: $\sum \cos(n\theta) J_n(x) = \cos(x\sin\theta)$

Integrate: $\int_0^{2\pi} \cos(x\sin\theta) d\theta = \sum_{n=-\infty}^{\infty} J_n(x) \int_0^{2\pi} \cos(n\theta) d\theta$
 $= \underline{J_0(x) 2\pi}$

4. (a) Want to show $\widehat{(f * g)} = \sqrt{2\pi} \hat{f} \hat{g}$,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y) \hat{g}(y) dy$$

$$\begin{aligned} \Rightarrow \widehat{(f * g)}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) \hat{g}(y) dy e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) \hat{g}(y) e^{-iky} e^{-ik(x-y)} dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(y) e^{-iky} \int_{-\infty}^{\infty} f(x-y) e^{-ik(x-y)} dx dy \end{aligned}$$

let $u = x - y$:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(y) e^{-iky} \int_{-\infty}^{\infty} f(u) e^{-iku} du dy \\ &= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi} \hat{g}(k)] [\sqrt{2\pi} \hat{f}(k)] = \sqrt{2\pi} \hat{f} \hat{g}. \end{aligned}$$

(b) $\hat{u}(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx$

and $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{\partial u}{\partial x} e^{-ikx} \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik) \frac{\partial u}{\partial x} e^{-ikx} dx$

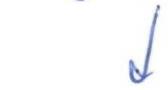
$= \frac{ik}{\sqrt{2\pi}} \left[(ik) u e^{-ikx} \right] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik) u e^{-ikx} dx$

$= \frac{ik}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-ikx} u \right] - \int_{-\infty}^{\infty} (-ik) u e^{-ikx} dx = -k^2 \hat{u}$

$\Rightarrow \nabla^2 u = 0$ becomes

$$\frac{\partial^2 \hat{u}}{\partial y^2} - k^2 \hat{u} = 0$$

$\hat{u} \rightarrow 0$ as $y \rightarrow \infty$
 $\hat{u}(k, 0) = \hat{f}(k)$.



$\Rightarrow \hat{u}(k, y) = A(k) \exp[-|k|y]$ since decaying in far field

$\rightarrow \hat{u}(k, y) = \hat{f}(k) \exp[-|k|y]$

$$\hat{u}(k,y) = \hat{f}(k) \exp[-|k|y]$$

Inverting, $u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-|k|y} e^{ikx} dk$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 \hat{f}(k) e^{ky} e^{ikx} dk + \int_0^{\infty} \hat{f}(k) e^{-ky} e^{ikx} dk \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y} \cos kx dk$$

Using convolution theorem with $g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} e^{-|k|y} \right] e^{ikx} dk$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|y} \cos kx dx$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-ky} \cos kx dx$$

$$= \frac{1}{\pi} \operatorname{Re} \left[\int_0^{\infty} e^{-ky + ikx} dx \right]$$

$$= \frac{1}{\pi} \operatorname{Re} \left[\frac{e^{-ky + ikx}}{ix - y} \right]_0^{\infty}$$

$$= \frac{1}{\pi} \operatorname{Re} \left[\frac{1}{y - ix} \right]$$

$$= \frac{1}{\pi} \frac{y}{y^2 + x^2}$$

$\Rightarrow u(x,y) = (f * g)$
 $= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2} ds$

∴

$$\begin{aligned}
 5(a) \quad \mathcal{L}[e^{-at}] &= \int_0^{\infty} e^{-st} e^{-at} dt \\
 &= \int_0^{\infty} e^{-(s+a)t} dt \\
 &= \left[\frac{1}{-(s+a)} e^{-(s+a)t} \right]_0^{\infty} \\
 &= \frac{1}{s+a}
 \end{aligned}$$

hence $\mathcal{L}[1] = \frac{1}{s}$

$$\begin{aligned}
 \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^{\infty} e^{-st} f'(t) dt \\
 &= \left[e^{-st} f(t) \right]_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= s\hat{f}(s) - f(0)
 \end{aligned}$$

(b) $\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = x$, $u(x, 0) = 1 + x^2$ $u(0, t) = 1$

↓ Laplace transform in t

$$s\bar{u}(s) - u(x, 0) + x \frac{\partial \bar{u}}{\partial x} = \frac{x}{s}$$

↓
 $\bar{u}(0, s) = \frac{1}{s}$

$$\rightarrow s\bar{u} + x \frac{\partial \bar{u}}{\partial x} = \frac{x}{s} + 1 + x^2$$

$$\rightarrow x \frac{\partial \bar{u}}{\partial x} + s\bar{u} = \frac{x}{s} + 1 + x^2$$

IF $\frac{dx}{x} = \frac{1}{s} dx = e^{\int \frac{1}{x} dx} = e^{s \ln x} = x^s$

$$\rightarrow \frac{\partial}{\partial x} (\bar{u} x^s) = \left(\frac{1}{s} + \frac{1}{x} + x \right) x^s = \frac{x^s}{s} + x^{s-1} + x^{s+1}$$

$$\bar{u} x^s = \frac{x^{s+1}}{s(s+1)} + \frac{x^s}{s} + \frac{x^{s+2}}{s+2} + A(s) D$$

$$\bar{u} = \frac{x}{s(s+1)} + \frac{1}{s} + \frac{x^2}{s+2} + \frac{D}{x^s}$$

Want $\bar{u}(0, s) = \frac{1}{s}$: $\bar{u}(0, s) = \frac{1}{s} + \lim_{x \rightarrow 0} x^{-s} D \Rightarrow D = 0$

$$\Rightarrow \bar{u}(x, s) = \frac{x^2}{s+2} + \frac{x}{s(s+1)} + \frac{1}{s} = \frac{x^2}{s+2} + \frac{x}{s} - \frac{x}{s+1} + \frac{1}{s}$$

$$\Rightarrow u(x, t) = x^2 e^{-2t} + x - x e^{-t} + 1$$

$$6. \quad u(x) + \int_{-\infty}^{\infty} f(x-\xi) u(\xi) d\xi = g(x)$$

$$\text{i.e. } u(x) + (f * u)(x) = g(x)$$

$$\Rightarrow \hat{u}(k) + \sqrt{2\pi} \hat{f}(k) \hat{u}(k) = \hat{g}(k)$$

$$\Rightarrow \hat{u}(k) = \frac{\hat{g}(k)}{1 + \sqrt{2\pi} \hat{f}(k)}$$

$$\Rightarrow u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k) e^{ikx}}{[1 + \sqrt{2\pi} \hat{f}(k)]} dk$$

$$\begin{aligned}
 f(x) = \frac{1}{2} e^{-|x|} &\Rightarrow \hat{f}(k) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \\
 &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos kx dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} \cos kx dx \\
 &= \text{Re} \left[\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x+ikx} dx \right] \\
 &= \frac{1}{\sqrt{2\pi}} \text{Re} \left[\frac{e^{-x+ikx}}{ik-1} \right]_0^{\infty} \\
 &= \frac{1}{\sqrt{2\pi}} \text{Re} \left[\frac{1}{ik-1} \right] = \frac{1}{\sqrt{2\pi} (1+k^2)}
 \end{aligned}$$

$$\begin{aligned}
 g(x) = e^{-x^2/2} &\Rightarrow \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ikx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos kx dx \\
 &= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} e^{-k^2/2} \quad (\text{by identity given}) \\
 &= e^{-k^2/2}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow u(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-k^2/2} e^{ikx}}{1 + \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi} (1+k^2)} \right)} dk \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-k^2/2 + ikx}}{2 + k^2} dk
 \end{aligned}$$

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